Scattering at a Junction of Two Waveguides with Different Surface Impedances

CORRADO DRAGONE, SENIOR MEMBER, IEEE

***Abstract*—We consider a junction of two cylindrical waveguides and derive the scattering matrix when a single mode is incident in one of the two waveguides. We are interested primarily in the case of two corrugated waveguides with different longitudinal impedances, but the analysis applies also to waveguides with nonzero transverse impedances. It is shown that, under certain general conditions, the infinite set of equations specifying the junction scattering coefficients can be solved exactly by the residue-calculus method. Very simple expressions are then obtained between the scattering coefficients and the propagation constants γ_n and γ_i of the modes in the two waveguides. These expressions, obtained previously only in special cases, are direct consequences of certain simple relations derived here for the coupling coefficients between the modes of the two waveguides. In those cases in which the scattering coefficients cannot be determined exactly, we determine them approximately by a perturbation analysis.**

I. Introduction

We CONSIDER a cylindrical boundary parallel to the z-axis and assume that the two regions $z>0$ and $z<0$ are characterized by different boundary conditions. In either region, it is assumed that the surface impedances for the longitudinal and transverse currents on the walls are given parameters independent of z. This accurately represents, for instance, a junction [1], [2] between two corrugated waveguides with a large number of corrugations per wavelength. The scattering matrix for this junction, when a mode is incident from one of the two waveguides, was derived in a previous article [1] by a perturbation analysis, assuming the difference in surface reactance between the two waveguides is small. Here, we remove this restriction and show that if only one mode can propagate in each corrugated waveguide then the reflection coefficient ρ_1 for this mode has its magnitude given *exactly* by the same expression derived in [1].

The treatment applies not only to corrugated waveguides, but in general also to waveguides with nonzero surface impedances in the longitudinal direction. If the difference between the boundary conditions of two waveguides is small, then the junction scattering coefficients can be derived straightforwardly by a perturbation analysis similar to the one of [1]. If the difference is not small, then in most cases of practical interest the scattering coefficients can be derived exactly as in [3], [4] by the residue-calculus method.

In fact, it is shown that, under certain conditions, the junction considered here is described by the same equations obtained previously for certain other discontinuities [3]–[5]. This equivalence is a consequence of certain simple relations derived in Section II for the coupling coefficients between the various modes of the two waveguides. Because of these relations, a simple solution is found to exist for the scattering coefficients, which are then simply related to the longitudinal wavenumbers γ_n and γ_i' of the various modes in the two waveguides. One thus obtains the same expressions derived in [6] for a parallel-plate waveguide and in [7] for a circular junction between a smooth and a corrugated waveguide, treated in [7] by the Wiener–Hopf technique. The results of this article are needed to determine accurately the input reflection ρ_1 of a corrugated feed [1], [8], and to design a suitable transformer to minimize ρ_1 as in [9].

Manuscript received November 24, 1983; revised April 20, 1984.
The author is with Bell Laboratories, Crawford Hill Laboratory, Holmdel, NJ 07733.

Geometric Interpretation of a Junction in Hilbert Space

This theory can be summarized as follows. In general, inside a waveguide, the field can be represented as a sum of modes, each characterized by $\partial/\partial z = -\gamma$, where γ denotes the longitudinal wavenumber of the mode. In the Hilbert space Π of the problem under consideration, each mode specifies a direction. Thus, a complete set of modes satisfying given boundary conditions specifies a coordinate system, i.e., a reference frame, whose orientation in Π is determined by the boundary conditions. In Section III, we find how the orientation of a frame is affected by the boundary conditions. Thus, we consider in general two modes, satisfying different boundary conditions. Their scalar product in Π is determined by a surface integral over the region S occupied by the waveguide in the x, y-plane. We show that this integral, given by the expression $(\boldsymbol{e}, \boldsymbol{h}')+(\boldsymbol{e}', \boldsymbol{h})$ of (15), can be reduced to the form

$$\frac{D}{\gamma'-\gamma}$$

where D is a contour integral which, under certain conditions, is separable into a product of two factors, each of which is determined by one of the two modes. Then, if the two modes have indexes n and i, respectively

$$\frac{D}{\gamma'-\gamma} \rightarrow \frac{D_n D_i'}{\gamma_i'-\gamma_n}$$

and an exact solution for the junction problem is readily obtained, for the following reason.

The modes that propagate in a waveguide can in general be divided into two groups, propagating in opposite direc-

0018-9480/84/1000-1319$01.00 ©1984 IEEE

tions. Therefore, the Hilbert space Π can be decomposed into two subspaces Π_+ and Π_-, of the same dimensionality, corresponding to the two directions of propagation. Negative labels will be used throughout the article for the modes propagating in the negative z-direction. The above decomposition obviously depends on the orientation of the particular coordinate system under consideration. That is, a rotation applied to the reference frame will change the decomposition. Now, the electromagnetic field at a junction (for $z=0$) between two waveguides can in general be represented by a vector $\mathscr{E}$ in Π. Since the two waveguides have different boundary conditions, their modes give rise to different decompositions of Π. Let Π_+, Π_- and Π'_+, Π'_- denote the two decompositions, with the prime denoting the waveguide for $z>0$. In the problem considered here, the projections of $\mathscr{E}$ onto Π_+ and Π'_- are given, since they represent the modes propagating towards the junction. The problem is to determine the projections of $\mathscr{E}$ onto Π_- and Π_+, which represent the scattered modes. In general, since $\Pi_+, \Pi_-, \Pi'_+, \Pi'_-$ have the same dimensionalities, the two given projections of $\mathscr{E}$ are sufficient to determine $\mathscr{E}$. Here it is assumed that the junction is excited from $z<0$ and, therefore, $\mathscr{E}'_- = 0$. Then, by requiring that the component of $\mathscr{E}$ with respect to the ith mode of Π'_- be zero, one obtains (38), whose solution is well known in the important case where $D_{n,i}$ are separable. One then obtains (50), assuming that only one mode is incident from $z<0$.

In this article, the above results are derived in a very general way, without a detailed knowledge of the properties of the two waveguides, or of the shape of their boundary. Of greatest interest in practice [8]–[10] is the problem of a waveguide filled with homogeneous material, but the results of Sections III–V apply in general, without this restriction. In particular, they are useful in the study of optical fibers, as shown in a future article. In Section IV, a very simple solution of (38) is derived in the important case where the two waveguides boundary conditions are only slightly different. Then the angle of rotation Θ between the two reference frames determined in Π by the modes of the two waveguides is small, and $\mathscr{E}$ can be determined by the following argument.

Since Θ is small, the difference $\delta\mathscr{E}$ between $\mathscr{E}'_-$ and $\mathscr{E}_-$ is small.[1] Furthermore

$$\delta\mathscr{E} = \mathscr{E}'_- - (\mathscr{E} - \mathscr{E}_+)$$

whose projections onto Π_+, Π'_- are

$$\delta\mathscr{E}_+ = (\mathscr{E}'_-)_+ \qquad \delta\mathscr{E}'_- = (\mathscr{E}_+)'_-.$$

But $\delta\mathscr{E}'_- \simeq \delta\mathscr{E}_-$, with error of order two in Θ, and we conclude that

$$\mathscr{E} \simeq \mathscr{E}_+ + \mathscr{E}'_- - (\mathscr{E}_+)'_- - (\mathscr{E}'_-)_+.$$

In words, $\mathscr{E}$ can be derived with error of order two in Θ by simply removing from $\mathscr{E}_+ + \mathscr{E}'_-$ the projections of $\mathscr{E}_+, \mathscr{E}'_-$ in Π'_-, Π_+, respectively. In the particular case of interest here, $\mathscr{E}'_- = 0$, since only one mode is incident from $z<0$. Therefore, $\mathscr{E} \simeq \mathscr{E}_+ - (\mathscr{E}_+)'_-$, and the reflected and transmitted modes are simply given by

$$\mathscr{E}_- \simeq (\mathscr{E}_+)'_- \qquad \mathscr{E}_+ \simeq (\mathscr{E}_+)'_+ \tag{1}$$

from which one can derive straightforwardly the scattered modes; the result is (41), (42). Of special interest is the reflection coefficient ρ_1 for the incident mode. We shall see that, only for certain waveguides

$$|\rho_1| = \left|\frac{1-\gamma_1}{1+\gamma_1}\right|$$

and the required conditions will be given in Section V. The above perturbation analysis is important, not only for its simplicity, but also because an exact treatment is not possible in certain cases, for instance when all the walls of a rectangular waveguide are corrugated as in [10].

[1]Similar considerations apply to $\mathscr{E}'_+ - \mathscr{E}_+$.

II. Boundary Conditions

The geometry of the problem is illustrated in Fig. 1. Two cylindrical waveguides are joined at $z=0$ along the closed curve C defining the cylindrical boundary of either waveguide. At the boundary, we introduce unit vectors $\boldsymbol{\nu}$ and $\boldsymbol{\tau}$ representing, respectively, the outwardly directed normal and the tangent, given by

$$\boldsymbol{\tau} = \boldsymbol{i}_z \times \boldsymbol{\nu}$$

$\boldsymbol{i}_z$ being a unit vector in the z-direction. Inside the boundary, the medium is assumed to be independent of z, but it is otherwise arbitrary. The boundary conditions for $z<0$ are assumed to be in the familiar form

$$E_\tau = jXH_z \qquad H_\tau = -jYE_z \tag{2}$$

involving the tangential field components in the τ, z-directions. The two parameters X and Y specify the surface impedances E_τ/H_z and $-E_z/H_\tau$ at the boundary. They are assumed to be different for the two waveguides, and a prime will be used to designate their values for $z>0$.

It is sometimes convenient to modify the definition of X, Y as in [1] by replacing X, Y in (2) with $XZ_0, Y/Z_0$, where $Z_0 = \sqrt{\mu_0/\epsilon_0}$. Then, the same substitution must be applied throughout this article.

III. Coupling Coefficient Between Two Modes

Consider two modes $\boldsymbol{E}, \boldsymbol{H}$ and $\boldsymbol{E}', \boldsymbol{H}'$ and let

$$\boldsymbol{e}(x,y)e^{-\gamma z} \qquad \boldsymbol{h}(x,y)e^{-\gamma z} \tag{3}$$

be the transverse parts of $\boldsymbol{E}, \boldsymbol{H}$. Let

$$e_z(x,y)e^{-\gamma z} \qquad h_z(x,y)e^{-\gamma z} \tag{4}$$

denote E_z, H_z and let a similar notation be used for $\boldsymbol{E}', \boldsymbol{H}'$. In the following section, we shall see that the equations specifying the amplitudes of the scattered modes in Fig. 1 have coefficients given by the expression

$$(\boldsymbol{e}, \boldsymbol{h}') + (\boldsymbol{e}', \boldsymbol{h}) \tag{5}$$

where the notation $(\boldsymbol{e}, \boldsymbol{h}')$ is used as in [1] to denote, over

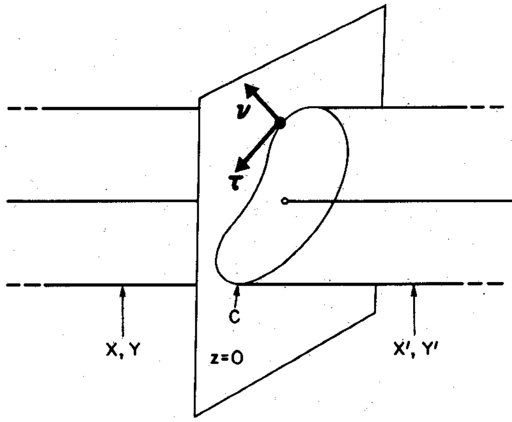


Fig. 1. Junction of two waveguides with different surface impedances.

the junction area S, the integral

$$(\boldsymbol{e},\boldsymbol{h}')=\iint_S(\boldsymbol{e}\times\boldsymbol{h}')\cdot\boldsymbol{i}_z\,dx\,dy. \tag{6}$$

In [1], this surface integral was reduced to a contour integral assuming real values of γ^2 and γ'^2, but the same derivation applies for complex values as well. In fact, the two modes satisfy the condition [4]

$$\nabla\cdot(\boldsymbol{E}\times\boldsymbol{H}'-\boldsymbol{E}'\times\boldsymbol{H})=0. \tag{7}$$

Decomposing the operator ∇ into its transverse and longitudinal components

$$\nabla=\nabla_t+(\gamma+\gamma')\boldsymbol{i}_z \tag{8}$$

integrating (7) over S and then using the divergence theorem, one obtains as in [1]

$$(\boldsymbol{e},\boldsymbol{h}')-(\boldsymbol{e}',\boldsymbol{h})=\frac{C}{\gamma+\gamma'} \tag{9}$$

where C is the sum of two integrals

$$C=M-N \tag{10}$$

where

$$M=\oint_C\left(e_\tau h_z'-e_\tau' h_z\right)d\tau \tag{11}$$

$$N=\oint_C\left(e_z h_\tau'-e_z' h_\tau\right)d\tau. \tag{12}$$

An important property of these two integrals is obtained by reversing the direction of propagation of one of the two modes. For instance, let the mode with wavenumber γ be replaced in (9)–(12) by the mode characterized by wavenumber $-\gamma$ and transverse field distributions

$$\boldsymbol{e}(x,y)e^{\gamma z}\qquad -\boldsymbol{h}(x,y)e^{\gamma z}. \tag{13}$$

The new mode is obtained by the substitution

$$\gamma,\boldsymbol{e},\boldsymbol{h},e_z,h_z\rightarrow-\gamma,\boldsymbol{e},-\boldsymbol{h},-e_z,h_z \tag{14}$$

which changes the sign of N without affecting M. Thus, (9) is changed into

$$(\boldsymbol{e},\boldsymbol{h}')+(\boldsymbol{e}',\boldsymbol{h})=\frac{D}{\gamma'-\gamma} \tag{15}$$

where

$$D=M+N. \tag{16}$$

Using (9), (15), one can determine $(\boldsymbol{e},\boldsymbol{h}')$

$$(\boldsymbol{e},\boldsymbol{h}')=\frac{\gamma' M+\gamma N}{\gamma'^2-\gamma^2}. \tag{17}$$

These basic expressions are direct consequences of (7). They only assume that the two modes satisfy Maxwell's equations with wavenumbers γ,γ' in the z-direction.

We now let $\boldsymbol{e}$ be the nth mode obtained in Fig. 1 for $z<0$, and let $\boldsymbol{h}'$ be the ith mode for $z>0$. Then, taking into account that e_τ, h_τ in (11), (12) must satisfy the boundary conditions (2), we obtain

$$(\boldsymbol{e}_n,\boldsymbol{h}_i')\mp(\boldsymbol{e}_i',\boldsymbol{h}_n)=\frac{M_{n,i}\mp N_{n,i}}{\gamma_i'\pm\gamma_n} \tag{18}$$

where

$$M_{n,i}=-j\oint_C(X'-X)h_{zn}h_{zi}'\,d\tau \tag{19}$$

$$N_{n,i}=-j\oint_C(Y'-Y)e_{zn}e_{zi}'\,d\tau. \tag{20}$$

We notice that

$$M_{n,i}=N_{n,i}=0,\qquad \text{for } X'=X, Y'=Y \tag{21}$$

and therefore, from (18), (20) by letting $X',Y'\rightarrow X,Y$, we obtain the orthogonality relations

$$(\boldsymbol{e}_n,\boldsymbol{h}_i)=0,\qquad \text{if } \gamma_n\neq\pm\gamma_i. \tag{22}$$

If X, Y and X',Y' are independent of τ, then $X'-X$ and $Y'-Y$ can be taken out of the integral signs in (19), (20) and, if there is no degeneracy, from (18), (20) we obtain

$$2j(\boldsymbol{e}_n,\boldsymbol{h}_n)=\left(\frac{\partial\gamma_n}{\partial X}\right)^{-1}\oint_C h_{zn}^2\,d\tau=\left(\frac{\partial\gamma_n}{\partial Y}\right)^{-1}\oint_C e_{zn}^2\,d\tau \tag{23}$$

which implies a simple relation between the partial derivatives of γ_n.

Notice, if a particular wavenumber γ is degenerate, the corresponding modes are not uniquely defined. Then, two independent modes are not necessarily characterized by $(\boldsymbol{e}_n,\boldsymbol{h}_i)=0$. However, all modes belonging to a degenerate γ can always be represented in terms of a set of modes satisfying the condition

$$(\boldsymbol{e}_n,\boldsymbol{h}_i)=0,\qquad \text{for } n\neq i \tag{24}$$

which will be assumed from now on. Then, one can show (see Appendix I) that

$$(\boldsymbol{e}_n,\boldsymbol{h}_n)\neq 0. \tag{25}$$

Taking this into account, one finds

$$(\boldsymbol{e}_n,\boldsymbol{h}_i)+(\boldsymbol{h}_n,\boldsymbol{e}_i)=0,\qquad \text{for } n\neq i. \tag{26}$$

Notice the condition $(\boldsymbol{e}_n,\boldsymbol{h}_i)=0$ is not satisfied for $\gamma_n=-\gamma_i$, as can be seen taking into account the transformation (14). Condition (26), on the other hand, is satisfied in all cases, including $\gamma_n=-\gamma_i$. Thus, the appropriate definition which should be used for the scalar product (or coupling coefficient) of two modes is given by (15), not (17), if both senses of propagation are considered, as in the next section.

If the medium is lossless, expressions similar to (15)–(23) can be derived by replacing $\boldsymbol{E}', \boldsymbol{H}'$ in (7) with the complex conjugates of $\boldsymbol{E}', -\boldsymbol{H}'$. One then obtains from (17) the coefficient $(\boldsymbol{e}_n, \boldsymbol{h}_i'^*)$ used in the perturbation analysis of [1]. Either one of the two coefficients can be used to determine the junction scattering coefficients (in the absence of losses, see Appendix I). However the derivation is simpler using $(\boldsymbol{e}_n, \boldsymbol{h}_i')$, because (17) involves γ_i' in place of $\gamma_i'^*$ appearing in the expression for $(\boldsymbol{e}_n, \boldsymbol{h}_i'^*)$. If the medium is lossy, $(\boldsymbol{e}_n, \boldsymbol{h}_i')$ is the appropriate coefficient, not $(\boldsymbol{e}_n, \boldsymbol{h}_i'^*)$. For $n = i$, the latter coefficient is needed to determine the power carried by the nth mode. However, it is shown in Appendix I that, if both the medium and the boundary are lossless, one can always choose the amplitude of each nondegenerate mode with imaginary γ so that both $\boldsymbol{e}(x, y)$ and $\boldsymbol{h}(x, y)$ are real, in which case $(\boldsymbol{e}, \boldsymbol{h})$ is real.

The relations derived so far are very general. In practice, of greatest interest are circular and rectangular waveguides. Then, in most cases the scattered modes have at the boundary the same τ-dependence of the incident mode. As a consequence, the coefficients $M_{n,i}$ dependence upon the indexes n, i is separable, so that $M_{n,i} = M_n M_i'$, and this is true also for $N_{n,i}$. For circular and rectangular waveguides, the wavenumbers γ and the coefficients M, N can be derived as shown in Appendix II. Notice that separable coefficients $M_{n,i}$ and $N_{n,i}$ do not necessarily imply separable $C_{n,i}$, since

$$C_{n,i} = C_n C_i' \tag{27}$$

requires an additional condition: $M_{n,i} = 0$, or $N_{n,i} = 0$, or $M_{n,i} = N_{n,i}$.

IV. Equations for the Scattering Coefficients

Now consider the field $\boldsymbol{E}, \boldsymbol{H}$ at the junction of Fig. 1 and assume that a single mode is incident from the left in Fig. 1. Let $\boldsymbol{E}_t, \boldsymbol{H}_t$ denote the transverse field components. To determine the amplitudes of the reflected and transmitted modes, we expand $\boldsymbol{E}_t$ and $\boldsymbol{H}_t$ on either side of the junction in an infinite series of modes, and then require continuity of $\boldsymbol{E}_t$ and $\boldsymbol{H}_t$ at the junction. For $z < 0$, representing $\boldsymbol{E}_t, \boldsymbol{H}_t$ in terms of the modes of the waveguide occupying the region $z < 0$

$$\boldsymbol{E}_t = A_1 \boldsymbol{e}_1 e^{-\gamma_1 z} + \sum_{n=-1}^{-\infty} A_n \boldsymbol{e}_n e^{-\gamma_n z} \tag{28}$$

$$\boldsymbol{H}_t = A_1 \boldsymbol{h}_1 e^{-\gamma_1 z} + \sum_{n=-1}^{-\infty} A_n \boldsymbol{h}_n e^{-\gamma_n z} \tag{29}$$

where

$$A_n \boldsymbol{e}_n e^{-\gamma_n z} \qquad A_n \boldsymbol{h}_n e^{-\gamma_n z}$$

are the transverse field components for the nth mode and $n = 1$ corresponds to the incident mode. Throughout the article, the wavenumbers γ_n with positive index will correspond to modes propagating in the positive z-direction, and we shall adopt the convention

$$\boldsymbol{e}_s = \boldsymbol{e}_{-s} \qquad \gamma_s = -\gamma_{-s} \tag{30}$$

which implies

$$\boldsymbol{h}_s = -\boldsymbol{h}_{-s}. \tag{31}$$

For $z > 0$, representing $\boldsymbol{E}_t, \boldsymbol{H}_t$ in terms of the modes occupying the region $z > 0$

$$\boldsymbol{E}_t = \sum_{i=1}^{\infty} A_i' \boldsymbol{e}_i' e^{-\gamma_i' z} \tag{32}$$

$$\boldsymbol{H}_t = \sum_{i=1}^{\infty} A_i' \boldsymbol{h}_i' e^{-\gamma_i' z}. \tag{33}$$

Taking into account the orthogonality relations (26), then (32), (33) give for $z = 0$

$$(\boldsymbol{E}_t, \boldsymbol{h}_i') = (\boldsymbol{e}_i', \boldsymbol{H}_t) = A_i'(\boldsymbol{e}_i', \boldsymbol{h}_i') \qquad (i > 0) \tag{34}$$

and therefore

$$(\boldsymbol{E}_t, \boldsymbol{h}_i') - (\boldsymbol{e}_i', \boldsymbol{H}_t) = 0 \qquad (i = 1, 2, \cdots) \tag{35}$$

$$(\boldsymbol{E}_t, \boldsymbol{h}_i') + (\boldsymbol{e}_i', \boldsymbol{H}_t) = 2A_i'(\boldsymbol{e}_i', \boldsymbol{h}_i') \qquad (i = 1, 2, \cdots). \tag{36}$$

The former condition is a direct consequence of the particular form of (32), (33), which does not contain modes with negative indexes, since the waveguide $z > 0$ is free of sources. Otherwise, if $\boldsymbol{E}, \boldsymbol{H}$ for $z > 0$ contained modes with negative indexes, one would have to replace (35) with

$$(\boldsymbol{E}_t, \boldsymbol{h}_i') - (\boldsymbol{e}_i', \boldsymbol{H}_t) = 2A_{-i}'(\boldsymbol{e}_i', \boldsymbol{h}_i') \qquad (i = 1, 2, \cdots) \tag{37}$$

whereas (36) would not be affected. Notice that either one of (36), (37) can be obtained from the other by replacing i with $-i$, taking into account that $\boldsymbol{e}_i' = \boldsymbol{e}_{-i}'$, $\boldsymbol{h}_i' = -h_{-i}'$.

Substituting (28), (29) in (35) we obtain

$$\sum_n A_{-n} \frac{D_{n,i}}{\gamma_i' - \gamma_n} = 0 \qquad (i = 1, 2, \cdots) \tag{38}$$

where $n = -1, 1, 2$, etc., and $D_{n,i}$ is obtained by applying (15) to the two modes $\boldsymbol{e}_n, \boldsymbol{h}_n$ and $\boldsymbol{e}_i', \boldsymbol{h}_i'$. Similarly, from (36)

$$\sum_n A_{-n} \frac{C_{n,i}}{\gamma_i' + \gamma_n} = 2A_i'(\boldsymbol{e}_i', \boldsymbol{h}_i') \qquad (i = 1, 2, \cdots) \tag{39}$$

where $n = -1, 1, 2$, etc., and

$$D_{n,i} = C_{-n,i} = M_{n,i} + N_{n,i}. \tag{40}$$

In the important case $M = 0$, the above expressions are analogous to those derived in [1], the main difference being here the appearance of γ_i' in place of $\gamma_i'^*$, as a consequence of the coefficient $(\boldsymbol{e}_n, \boldsymbol{h}_i')$ used here in place of $(\boldsymbol{e}_n, \boldsymbol{h}_i'^*)$.

Since the incident mode is assumed to be given, the coefficient A_1 in (38), (39) is known. The unknowns are the coefficients A_{-n}, A_i with positive n, i. In the following, a one-to-one correspondence will be assumed to exist between the modes in the two waveguides, and

$$\gamma_i' \to \gamma_i, \qquad \text{for } X', Y' \to X, Y.$$

Then, (38), (39) can be solved approximately by introducing a large constant N and neglecting all modes with indexes larger than N. From (38) for $i \leqslant N$, one then obtains N equations which can be solved for the N unknowns A_{-1}, A_{-2}, etc. Similarly, (39) can be solved for A'_n. We shall see in Section V that, by letting $N \to \infty$, one obtains in most cases the correct solution as expected. In certain cases, however, it may happen that the expressions obtained for $N \to \infty$ are afflicted by relative convergence (see [3], [6]), in which case the solution may not be unique and in order to obtain the correct solution and modes must be properly indexed as shown in [3].

Solution when $X' - X$ and $Y' - Y$ are Small

If both $X' - X$ and $Y' - Y$ are small then the incident field is only slightly perturbed by the discontinuities $\delta X' = X' - X$ and $\delta Y' = Y' - Y$ and (38), (39) can be solved as in [1], [8]. In fact, then all scattered amplitudes except A'_1 are small and, furthermore

$$C_{n,i} \simeq 0, \qquad \text{for } n \neq i$$

in view of the orthogonality relations (26). As a consequence, only two terms, those for $n = -1$, i need be considered in (38), and from (35) with $i = n$ one obtains

$$A_{-n} \simeq -A_1 \frac{(e_1, h'_n) - (e'_n, h_1)}{(e_n, h'_n) + (e'_n, h_n)}. \tag{41}$$

Similarly, from (36)

$$A'_i \simeq A_1 \frac{(e_1, h'_i) + (e'_i, h_1)}{2(e'_i, h'_i)}. \tag{42}$$

If $\delta X = 0$ or $\delta Y = 0$, then for $n = 1$

$$A_{-1} = \pm \frac{\gamma'_1 - \gamma_1}{\gamma'_1 + \gamma_1} A_1 \tag{43}$$

which should be compared with the exact expression derived in the following section, assuming $D_{n,i} = D_n D_1$. An important application of (41), (43) is obtained considering a rectangular waveguide with all four walls corrugated as in [10], since then the junction cannot be treated as in Section V. Then, an approximate calculation of the input reflection ρ_1 of a feed designed as in [9] can be carried out using (43), with γ'_1, γ_1 calculated using the asymptotic theory of [11]. Using (43), one can also minimize ρ_1 using a matching transformer as in [9].

If both δX and δY are nonzero, then using (41) one can verify that in general there is no simple relation between the reflected amplitude A_{-1} and the wavenumbers γ'_1, γ_1. This implies that the simple relations derived in the following section between the reflection and transmission coefficients ρ_n, t_i and the wavenumbers γ_n, γ'_i are valid only under the particular conditions of Section V.

V. Solution When $C_{n,i} = C_n C'_i$

We now assume, for all the scattered modes, that the tangential field components at the boundary have the same τ-dependence of the incident mode. If then either $X' = X$ or $Y' = Y$, one has from Section III that

$$C_{n,i} = C_n \cdot C'_i \qquad D_{n,i} = \pm C_{n,i} \tag{44}$$

and (38), (39) can be written in the form

$$\sum_n \frac{R_{-n}}{\gamma_n - \gamma'_i} = 0 \tag{45}$$

$$\sum_n \frac{R_{-n}}{\gamma_n + \gamma'_i} = T'_i \tag{46}$$

where $n = -1, 1, 2$, etc., and $R_n = A_n C_n$, $T'_i = \mp 2A'_i(e'_i, h'_i)/C'_i$. The above equations also apply to a junction characterized by $X = Y$ and $X' = Y'$, as pointed out in Appendix IV.

We now assume that the difference $\gamma'_m - \gamma_m$ approaches a finite limit for $m \to \infty$. Then, (45) and (46) can be solved as in [3] with the help of the two integrals

$$\frac{1}{2\pi j} \oint_D \frac{f(\omega)}{\omega \pm \gamma'_i} d\omega \tag{47}$$

where D is an infinitely large circle in the complex ω-plane and $f(\omega)$ is the function

$$f(\omega) = h \frac{1}{1 + \frac{\omega}{\gamma_1}} \prod_{m=1}^{\infty} \frac{\left(1 - \frac{\omega}{\gamma'_m}\right)}{\left(1 - \frac{\omega}{\gamma_m}\right)} \tag{48}$$

with h being a suitable constant.

Taking into account the assumed behavior of $\gamma'_m - \gamma_m$ for $m \to \infty$, one can determine as in [3] (see Appendix III) the behavior of $f(\omega)$ for $|\omega| \to \infty$, and one finds that the integrals in (47) vanish. Thus, expressing each integral in terms of the residues of $f(\omega)$ and equating the result to zero, one obtains (45) and (46) with

$$R_{-n} = Res f(\gamma_n) \qquad T'_n = -f(-\gamma'_n) \tag{49}$$

which give[2]

$$\rho_n = -\frac{R_{-n}}{R_1} = \frac{\gamma'_n - \gamma_n}{\gamma'_n + \gamma_1} \prod_{m=1}^{\infty}{}^{(n)} \frac{(1 - \gamma_n/\gamma'_m)(1 + \gamma_1/\gamma_m)}{(1 - \gamma_n/\gamma_m)(1 + \gamma_1/\gamma'_m)} \tag{50}$$

and

$$t_n = \frac{T'_n}{R_1} = \frac{1}{\gamma'_n - \gamma_1} \prod_{m=1}^{\infty} \frac{(1 + \gamma'_n/\gamma'_m)(1 + \gamma_1/\gamma_m)}{(1 + \gamma'_n/\gamma_m)(1 + \gamma_1/\gamma'_m)}. \tag{51}$$

VI. Discussion

As pointed out in the introduction, a mode e', h' specifies a direction in the Hilbert space Π. Of general interest in the theory of scattering and radiation in waveguides is the problem of determining the components of a vector $\mathscr{E}$ in the direction of a given mode. For the electromagnetic-field distribution produced by E, H over the plane $z = 0$,

[2]We use the notation $\Pi^{(n)}$ to indicate omission of the factor with $m = n$.

we have shown that the vector

$$\mathscr{E} = \begin{bmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{bmatrix}, \qquad \text{for } z = 0$$

has the component

$$\frac{(\boldsymbol{E}, \boldsymbol{h}') + (\boldsymbol{e}', \boldsymbol{H})}{2(\boldsymbol{e}', \boldsymbol{h}')} \begin{bmatrix} \boldsymbol{e}' \\ \boldsymbol{h}' \end{bmatrix} \tag{52}$$

in the direction of $\bar{e}', \boldsymbol{h}'$. In the special case where $\boldsymbol{E}, \boldsymbol{H}$ is characterized by $\partial/\partial z = -\gamma$ we have reduced the surface integrals $(\boldsymbol{E}, \boldsymbol{h}')$ and $(\boldsymbol{e}', \boldsymbol{H})$ to contour integrals. An important application arises in the theory of diffraction. If, for instance, the plane $z = 0$ is the aperture of a horn and $\boldsymbol{E}, \boldsymbol{H}$ represents one of the modes of the horn, then letting $\boldsymbol{e}', \boldsymbol{h}'$ be a plane wave, one obtains from (52) the far-field amplitude radiated in the direction of $\boldsymbol{e}', \boldsymbol{h}'$.

The coupling coefficient between two modes is given by

$$(\boldsymbol{e}, \boldsymbol{h}') + (\boldsymbol{e}', \boldsymbol{h})$$

which vanishes for $\gamma' \neq \gamma$ when the two modes satisfy the same boundary conditions. This important property applies even when either mode is obtained from the other through a reversal of the propagation direction as in (14), in which case $\gamma' = -\gamma$ and the coefficients $(\boldsymbol{e}, \boldsymbol{h}')$, $(\boldsymbol{e}', \boldsymbol{h})$ do not vanish. Direct consequences of the above property are (35), (36), from which (38), (39) directly followed.

If the two modes do not satisfy the same boundary conditions, then their coupling coefficient does not vanish and a simple relation, (18), exists between the above coefficient and the wavenumbers of the two modes. For certain waveguides, the coefficients D and C in (9), (15) are separable

$$C_{n,i} = C_n C_i'$$

and then an exact solution of (38), (39) was obtained in Section V. The above condition is satisfied in two important cases: in a circular waveguide with transverse corrugations causing $X' = X = 0$, and in a rectangular waveguide with corrugations on only two walls. The wavenumbers γ_n and γ_n' in these two cases can be derived as in Appendix II, where it is shown that if both Y and Y' are nonzero the difference $\gamma_n' - \gamma_n$ approaches a finite limit for $n \to \infty$. If, instead, either Y or Y' is zero, then the limit vanishes. In either case, one obtains the solution of Section V.

Of greatest interest is the reflection coefficient ρ_1 which can be minimized using a matching section as in [9]. If only the first s modes corresponding to $\gamma_1, \cdots, \gamma_s$ propagate, then in (50) for $n = 1$ the factors with $m > s$ have unit magnitude. In fact, it is shown in Appendix I that in general the cutoff modes can be divided into two groups: in one group all γ_n are real and, in the other, the modes can be ordered so that $\gamma_n = \gamma_{n+1}^*$. Taking this into account

$$|\rho_1| = \left| \frac{\gamma_1' - \gamma_1}{\gamma_1' + \gamma_1} \prod_{m=2}^{s} \frac{(1 - \gamma_1/\gamma_m')(1 + \gamma_1/\gamma_m)}{(1 - \gamma_1/\gamma_m)(1 + \gamma_1/\gamma_m')} \right|$$

which involves only the propagating modes (with imaginary γ_n). If only one mode propagates, as in [1], then

$$|\rho_1| = \left| \frac{\gamma_1' - \gamma_1}{\gamma_1' + \gamma_1} \right|$$

and therefore in this case the perturbation analysis of Section IV gives exactly the magnitude of ρ_1.

Of greatest importance in practice is the case where $X' - X = 0$ and $Y' - Y$ is small. Then, $\gamma_n' \simeq \gamma_n$ and from (50), (51) one obtains the expressions derived in Section IV or in [1]. Another important case arises when both Y and Y' are large. Then γ_m and γ_m' differ little from the values obtained for $Y, Y' \to \infty$, and they can be determined as shown in Appendix II.

Notice, in the case of a rectangular boundary, that the exact derivation of Section V does not apply to the waveguide of [10] with all four walls corrugated, since then the junction violates the condition $C_{n,i} = C_n C_i'$. The perturbation analysis of Section IV shows, however, that (50), (51) are approximately valid also in this case, provided $Y' - Y$ is small.

Appendix I

Coupling Between Conjugate Modes in a Lossless Waveguide

In the treatment of scattering or radiation in waveguides, the coupling coefficient between two modes is usually expressed in terms of an integral involving the components of one mode multiplied by the complex conjugate components of the other mode.

In our definition, instead, the coupling coefficient is given by the symmetric expression $(\boldsymbol{e}, \boldsymbol{h}') + (\boldsymbol{e}', \boldsymbol{h})$, which does not involve complex conjugate components. An objection which may be raised to this definition is that some of the modes may be characterized by $(\boldsymbol{e}, \boldsymbol{h}) = 0$, if γ is degenerate. In a circular waveguide, for instance, $(\boldsymbol{e}, \boldsymbol{h}) = 0$ for the modes with ϕ-dependence given by

$$e^{\pm jm\phi}.$$

However, these modes can always be expressed in terms of the modes with ϕ-dependence given by

$$\cos m\phi \qquad \sin m\phi$$

which satisfy the condition $(\boldsymbol{e}, \boldsymbol{h}) \neq 0$. One can show that in general, by requiring

$$(\boldsymbol{e}_n, \boldsymbol{h}_i) + (\boldsymbol{e}_i \boldsymbol{h}_n) = 0, \qquad \text{for } i \neq k$$

one guarantees $(\boldsymbol{e}_n, \boldsymbol{h}_n) \neq 0$.

If the medium is lossless, it was pointed out in Section III that expressions similar to (15)–(23) can be derived by replacing $\boldsymbol{E}', \boldsymbol{H}'$ in (7) with the complex conjugates of $\boldsymbol{E}', -\boldsymbol{H}'$. One then obtains from (17) the coefficient $(\boldsymbol{e}, \boldsymbol{h}'^*)$ used in [1]. In the perturbation analysis of [1], use of $(\boldsymbol{e}, \boldsymbol{h}'^*)$ in place of $(\boldsymbol{e}, \boldsymbol{h}')$ did not cause difficulties because consideration was restricted in [1] to the modes with real values of γ'^2 and γ^2. In the present article, since complex values of γ'^2 and γ^2 are considered, a more appropriate definition is given by $(\boldsymbol{e}, \boldsymbol{h}')$.

It was shown in [1] that

$$(\boldsymbol{e},\boldsymbol{h}'^*)=\frac{\gamma'^*}{\gamma'^{*2}-\gamma^2}\oint_c\left(e_\tau h_z'^*+e_\tau'^*h_z\right)d\tau+\frac{\gamma}{\gamma'^{*2}-\gamma^2}\oint_c\left(e_z h_\tau'^*+e_z'^*h_\tau\right)d\tau. \quad (53)$$

This basic result was derived in [1] assuming that both γ^2 and γ'^2 are real, but that derivation applies without modifications also for complex values of γ^2 and γ'^{*2}.

Equation (17) can be derived directly from (53) by applying to $\boldsymbol{E}',\boldsymbol{H}'$ the transformation

$$\boldsymbol{e}',\boldsymbol{h}',e_z',h_z',\gamma'\rightarrow\boldsymbol{e}'^*,-\boldsymbol{h}'^*,e_z'^*,-h_z'^*,\gamma'^*. \quad (54)$$

One can verify that the result, which will be called the conjugate mode of $\boldsymbol{E}',\boldsymbol{H}'$, satisfies Maxwell's equations.

Notice that if the boundary is lossless so that the parameters X',Y' are real, then the conjugate mode satisfies the same boundary conditions of $\boldsymbol{E}',\boldsymbol{H}'$. Then, in either one of the two waveguides of Fig. 1, the modes with complex values of γ'^2 (or γ^2) can be grouped as in Section VI into pairs of conjugate modes, related by the transformation (54).

Also notice that the propagation constant γ of a mode is not affected by the above transformation if γ is real. Thus, for a nondegenerate mode $\boldsymbol{e},\boldsymbol{h}$ with real γ, the transformation (54) is equivalent to multiplication of $\boldsymbol{e},\boldsymbol{h}$ by a constant factor A of unit amplitude. If instead γ is imaginary, the sign of γ is changed and the transformation (54) is equivalent to multiplication of $\boldsymbol{e},-\boldsymbol{h}$ by A. We conclude that by properly choosing the mode amplitude so that $A=1$ one will obtain

$$\boldsymbol{e}_n=\boldsymbol{e}_n^*,\ \boldsymbol{h}_n=\mp\boldsymbol{h}_n^*$$

depending on whether γ_n is real or imaginary. In the former case, the mode does not carry real power, since $(\boldsymbol{e},\boldsymbol{h}^*)$ is imaginary. In the latter case $(\boldsymbol{e},\boldsymbol{h})$ is real, and $(\boldsymbol{e},\boldsymbol{h})\neq 0$, as pointed out in Section III. If γ is complex

$$\left(\boldsymbol{e}_n,\boldsymbol{h}_n^*\right)=0$$

as can be seen from (53) by letting $X,Y\rightarrow X',Y'$ for $n=i$.

Appendix II

Circular and Rectangular Waveguides

As a first application of the results of Section V, consider in Fig. 2 a circular waveguide of radius a, let the boundary parameters X, Y and X',Y' be independent of ϕ, and assume the medium inside the boundary is homogeneous and lossless. Then, the coefficients $C_{n,i}$ are separable as in Section V if $X=X'$ or $Y=Y'$ or both $X=Y$ and $X'=Y'$. Here, we consider the special case

$$X=X'=0$$

which corresponds to a corrugated waveguide, and assume E_z for the incident mode has ϕ-dependence given by $\cos\phi$.

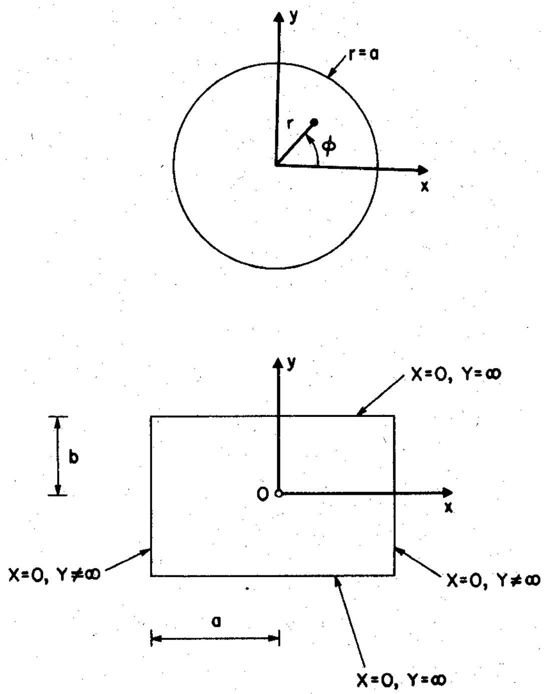


Fig. 2. Rectangular and circular boundaries.

Then, for each mode for $z<0$ one has

$$E_z=cJ_1\left(u\frac{r}{a}\right)\cos\phi e^{-\gamma z} \quad (55)$$

$$ZH_z=dJ_1\left(u\frac{r}{a}\right)\sin\phi e^{-\gamma z} \quad (56)$$

where $Z=\sqrt{\frac{\mu}{\epsilon}}$, $(\gamma a)^2=u^2-(ka)^2$, and $r^2=x^2+y^2$. Then (10), (19), and (20) for $X'-X=0$ give

$$C_{n,i}=-N_{n,i}=j\pi a(Y'-Y)\alpha_n\alpha_i' \quad (57)$$

where α_n is determined by the nth values of c and u

$$\alpha_n=c_nJ_1(u_n)$$

and similarly for α_i'. The characteristic equations which determine u_n and c_n/d_n can be written in the form [1], [2]

$$G=j\frac{u^2}{\gamma a}\left(YZ-\frac{ka}{u}\frac{J_1'}{J_1}\right)$$

$$\frac{1}{G}=j\frac{u^2}{\gamma a}\left(\frac{X}{Z}-\frac{ka}{u}\frac{J_1'}{J_1}\right)$$

where $G=d/c$. Notice here we are assuming $X=0$.

We now derive u_m when either m or Y are large and show that the difference $\gamma_m'-\gamma_m$ approaches a finite limit for $m\rightarrow\infty$ as assumed in Section V. From the above relations for $X=0$

$$G=-jF(u)\frac{ka}{\gamma a} \quad (58)$$

where

$$F(u)=u\frac{J_1'(u)}{J_1(u)} \quad (59)$$

and

$$(\gamma a)^2 = u^2 - (ka)^2. \tag{60}$$

Furthermore, the characteristic equation which determines the eigenvalues u_n can be written in the form

$$T_\pm(u) = 2F(u) \tag{61}$$

where

$$T_\pm(u) = \frac{YZ}{ka}u^2 \pm \sqrt{\left(\frac{YZ}{ka}u^2\right)^2 + 4\frac{(ka)^2 - u^2}{(ka)^2}}. \tag{62}$$

For

$$|YZ| \geqslant \frac{1}{ka} \tag{63}$$

$T_\pm(u)$ is a real valued function of u and the eigenvalues u_n are all real. For $YZu^2/ka \to \infty$

$$T_- \to 2\left(\frac{YZ}{ka}u^2\right)^{-1}\left[\left(\frac{u}{ka}\right)^2 - 1\right] \tag{64}$$

and

$$T_+ \to 2\frac{Y}{ka}u^2. \tag{65}$$

Therefore, the eigenvalues determined by T_- are given by

$$u_m \simeq a_m\left[1 + \frac{(ka)^2 - a_m^2}{YZkaa_m^2}\frac{1}{(a_m^2 - 1)}\right] \tag{66}$$

where a_m is one of the roots of $J_1'(u)$. Since $G \simeq 0$, the corresponding modes are approximately of the TE type. For the modes determined by T_+

$$u_m \simeq b_m\left(1 + \frac{ka}{YZb_m^2}\right) \tag{67}$$

where b_m is one of the roots of $J_1(u)$ and, since $G \simeq \infty$, these modes are of the TM type. Using the above expressions, one can determine straightforwardly the eigenvalues u_m if Y is large, or if Y is finite but u_m is large. In the latter case of large u_m, taking into account the asymptotic behavior of a_m and b_m for large m, (66) and (67) give

$$u_m \simeq m\frac{\pi}{2} + \frac{\pi}{4} \qquad (m = 1, 2, \cdots) \tag{68}$$

which for m odd gives the roots of $J_1'(u)$ and for m even, the roots of $J_1(u)$.

If

$$|Y| \leqslant \frac{1}{ka} \tag{69}$$

then $T_\pm(u)$ becomes complex for $\xi_1^2 < u^2 < \xi_2^2$, where

$$\xi_{1,2}^2 = \frac{2}{(YZ)^2}\left[1 \mp \sqrt{1 - (Yka)^2}\right] \tag{70}$$

since the radical in (62) vanishes for $u^2 = \xi_i^2$. For $YZ \simeq 0$

$$\xi_1 \simeq (ka)^2 \qquad \xi_2 \simeq \infty \tag{71}$$

in which case the real solutions of (61) are characterized by $u < ka$.

Finally, consider the case $Y = 0$. Then, from (58)–(62) one has $G = \pm 1$ and

$$-\frac{kau}{\sqrt{(ka)^2 - u^2}}\frac{J_1'(u)}{J_1(u)} = \pm 1. \tag{72}$$

For large values of u

$$\frac{J_1'}{J_1} \to -\tan\left(u - \frac{3}{4}\pi\right) \tag{73}$$

and therefore (72) gives

$$u_m \to \frac{3}{4}\pi + m\pi \pm j\tanh^{-1}\left(\frac{1}{ka}\right). \tag{74}$$

Rectangular Waveguide

As a second application, consider a hollow rectangular waveguide. Then, the modes can be expanded in terms of a finite sum of elementary functions only if the values of X, Y satisfy suitable conditions[3] derived by Dydbal *et al.* [12]. Here, consideration will be restricted to the case where $X = 1/Y = 0$ for the two walls orthogonal to the y-axis (Fig. 2). For the other two walls, we assume $X = 0$ and

$$\frac{1}{Y} \neq 0$$

with Y independent of x, y. This accurately represents a waveguide in which only two walls, those orthogonal to the x-axis, are corrugated.

Under the above conditions, the modes can be divided into two groups, characterized, respectively, by $E_y = 0$ and $H_y = 0$. For the latter modes, one can show that the field is not affected by the value of Y, and therefore only the former modes will be considered. For the even modes, one can write

$$E_x = c\cos\left(\frac{ux}{a}\right)\cos(ty)e^{-\gamma z} \tag{75}$$

$$E_z = -\frac{u}{a\gamma}c\sin\left(\frac{ux}{a}\right)\cos(ty)e^{-\gamma z} \tag{76}$$

where $t = (2s+1)\pi/2b$ and $2a$, $2b$ are the waveguide dimensions in the x, y-directions. One then finds that $C_{n,i}$ is given by (57) with πa replaced by $2b$ and $\alpha_n = c_n \cos u_n$. The characteristic equation which specifies the eigenvalues can be written in the form

$$u\tan u = \xi \tag{77}$$

where

$$\xi = \frac{a}{b}\frac{(kb)^2 - (tb)^2}{YZkb}. \tag{78}$$

If $\xi = 0$, then

$$u_m = (m-1)\pi \qquad (m = 1, 2, \cdots) \tag{79}$$

which also gives the asymptotic behavior for u_m for large

[3]When these conditions are not satisfied, as in [10], the modes may be derived by the approximate procedure of [11].

m when $\xi \neq 0$. If $\xi = \infty$, then

$$u_m = (2m-1)\frac{\pi}{2} \cdot (m=1,2,\cdots). \tag{80}$$

Notice that all u_m are real if $\xi \geqslant 0$. If $\xi < 0$, the eigenvalue corresponding to $m=1$ in (79) is imaginary. In Section IV, the index $m=1$ was assigned to the incident mode, whereas (79) assigns the index $m=1$ to the mode with $u=0$ for $\xi = 0$. Thus, the indices of these two modes may have to be interchanged, in order to conform to the convention of Section IV. If ξ is small, then from (77) for $m \neq 1$

$$u_m \simeq (m-1)\pi\left(1+\frac{1}{\xi}\frac{1}{(m-1)^2\pi^2}\right). \tag{81}$$

For $m=1$, $u_1^2 \simeq 1/\xi$.

Appendix III

Asymptotic Behavior of $f(\omega)$ for $\omega \to \infty$

First consider the case where both Y and Y' are nonzero. Then, taking into account the behavior of u_m for $m \to \infty$, one has

$$\gamma_m' - \gamma_m \to 0, \qquad \text{for } m \to \infty$$

and from [3] this implies for large ω

$$f(\omega) \to \frac{1}{\omega}.$$

If instead either Y or Y' is zero then $\gamma_m' - \gamma_m$ approach a nonzero limit for $m \to \infty$ and from [3]

$$f(\omega) \to \frac{1}{\omega\sqrt{\omega}}.$$

Appendix IV

A hollow waveguide satisfying the condition

$$\frac{X}{Y} = Z^2 \qquad Z^2 = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

has certain interesting properties [13] which are direct consequence of the invariance of Maxwell's equations to the substitution

$$\boldsymbol{E} \to Z\boldsymbol{H} \qquad Z\boldsymbol{H} \to -\boldsymbol{E}. \tag{82}$$

In general, if $X \neq YZ^2$, this substitution changes the waveguide boundary conditions (2) according to the transformation $X \to YZ^2$, $YZ^2 \to X$, but it does not affect a waveguide with $X = YZ^2$. Such a waveguide, is degenerate, since each mode is in general transformed by (82) into a different mode with the same propagation constant. Thus, if $\boldsymbol{E} = \boldsymbol{A}$, $Z\boldsymbol{H} = \boldsymbol{B}$ is a particular solution of Maxwell's equations in such a waveguide, then also

$$\boldsymbol{E} = \boldsymbol{A} - \alpha\boldsymbol{B} \qquad Z\boldsymbol{H} = \alpha\boldsymbol{A} + \boldsymbol{B} \tag{83}$$

is a solution, containing an arbitrary parameter α. Clearly, all solutions can be divided into two groups, obtained from (83), respectively, for $\alpha = j$ and $\alpha = -j$. If the transformation (82) is applied to either group, one finds that the result is simply multiplication by $+j$ or $-j$, depending on whether $\alpha = j$ or $-j$. These considerations apply, not only to a cylindrical waveguide, but in general to any structure whose boundary conditions are invariant to the transformation (82). Thus, it is interesting to examine the far-field behavior of a horn having such boundary conditions. In the far-field, $Z\boldsymbol{H} = \boldsymbol{i}_r \times \boldsymbol{E}$, and the transformation (82) is equivalent to $\boldsymbol{E} \to \boldsymbol{i}_r \times \boldsymbol{E}$, $Z\boldsymbol{H} \to -\boldsymbol{i}_r \times \boldsymbol{E}$, simply producing a 90° rotation of $\boldsymbol{E}$, $Z\boldsymbol{H}$. Furthermore, for $\alpha = \pm j$ the far-field is everywhere circularly polarized.

Now consider a junction between two waveguides with $X = YZ^2$ and $X' = Y'Z^2$ and let $\alpha = j$. Then, consideration can be restricted to the modes satisfying the condition

$$\boldsymbol{e}_n = jZ\boldsymbol{h}_n \qquad \boldsymbol{e}_i' = jZ\boldsymbol{h}_i'. \tag{84}$$

But according to (30) and (31)

$$\boldsymbol{e}_n = \boldsymbol{e}_{-n} \qquad \boldsymbol{h}_n = -\boldsymbol{h}_{-n} \tag{85}$$

which implies the following. Let the indexes for the modes in the two waveguides be chosen so that $\boldsymbol{e}_i \to \boldsymbol{e}_i'$, $\boldsymbol{h}_i \to \boldsymbol{h}_i'$ for $X', Y' \to X, Y$ and consider a particular mode produced for $z > 0$. Then $\boldsymbol{e}_i', \boldsymbol{h}_i'$ satisfies condition (84) but, because of (85), condition (84) is violated by the corresponding reflected mode $\boldsymbol{e}_n, \boldsymbol{h}_n$ with $n = -i$. Thus, the corresponding reflection coefficient is zero; in particular, $\rho_1 = 0$. Next, consider one of the modes $\boldsymbol{e}_i', \boldsymbol{h}_i'$ which violate condition (84). Such a mode is not excited for $z > 0$, but the corresponding mode $\boldsymbol{e}_n, \boldsymbol{h}_n$ with $n = -i$ is reflected for $z < 0$, because it satisfies condition (80).

Finally, consider a junction between two circular waveguides with $X = YZ^2$, $X' = Y'Z^2$. Then, in (38) and (39) for $n > 0$ one has either

$$D_{n,i} = 0$$

or

$$D_{n,i} = 2M_{n,i} = 2M_n M_i'.$$

The first case arises when $\boldsymbol{e}_n$ and $\boldsymbol{e}_i'$ have the same α and, the latter case, when they have different α. The opposite is true for $n < 0$. Taking this into account one finds that (38) and (39) can be reduced to the form of (45) and (46).

References

[1] C. Dragone, "Reflection, transmission and mode conversion in a corrugated feed," *Bell Syst. Tech. J.*, vol. 56, no. 6, pp. 835–868, July–Aug. 1977.

[2] P. J. B. Clarricoats and P. K. Saha, "Propagation and radiation behavior of corrugated feeds; Part 1—Corrugated waveguide feed," *Proc. Inst. Elec. Eng.*, vol. 118, no. 9, pp. 1167–1176, Sept. 1971.

[3] R. Mittra and S. W. Lee, *Analytical Technique in the Theory of Guided Waves.* New York: MacMillian, 1971.

[4] R. E. Collin. *Field Theory of Guided Waves.* New York: McGraw-Hill, 1960.

[5] R. A. Hurd and H. Gruenberg, "H-plane bifurcation of rectangular waveguides," *Can. J. Phys.*, vol. 32, pp. 694–701, 1954.

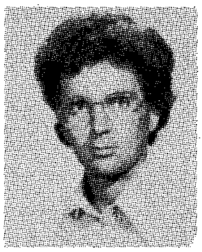
[6] D. S. Karjala and R. Mittra, "Scattering of the junction of two semi-infinite parallel impedance plane waveguides," *Can. J. Phys.*, vol. 43, no. 5, pp. 849–854, 1965.

[7] V. G. Daniele, I. Montrosset, and R. S. Zich, "Wiener–Hopf solution for the junction between a smooth and a corrugated cylindrical waveguide," *Radio Sci.*, vol. 14, no. 6, pp. 943–955, Nov.–Dec. 1979.

[8] C. Dragone, "Characteristics of a broadband microwave corrugated feed: A comparison between theory and experiment," *Bell Syst. Tech. J.*, vol. 56, no. 6 pp. 869–888, July–Aug. 1977.

[9] C. Dragone and W. E. Legg, "Use of a quarter-wave transformer to

reduce the input reflection of a corrugated feed," *Bell Syst. Tech. J.* Feb. 1984.
[10] C. Dragone, "A rectangular horn of four corrugated plates," submitted to *IEEE Trans. Antennas Propagat.*
[11] C. Dragone, "High-frequency behavior of waveguides with finite surface impedances," *Bell Syst. Tech. J.*, vol. 60, no. 1, pp. 89–115, Jan. 1981.
[12] V. L. Rumsey, "Horn antennas with uniform power patterns around their axes," *IEEE Trans. Antennas Propagat.*, vol. AP-14, p. 656, Sept. 1966.
[13] R. B. Dydbal, L. Peters, and W. H. Peake, "Rectangular waveguide and impedance walls," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-19, no. 1, pp. 2–9, Jan. 1971.



Corrado Dragone (SM'84) received the Laurea in electrical engineering from Padua University, Italy, in 1961 and the Libera Docenza from the Ministero della Pubblica Istruzione, Italy, in 1968.

Since joining Bell Laboratories in 1961, he has been engaged in experimental and theoretical work on microwave antennas and solid-state power sources. In 1974 and 1975, he taught a course on antennas and propagation at Padua University. He is currently concerned with problems involving electromagnetic wave propagation and microwave antennas for terrestrial radio systems and satellite communications.

High-Temperature Microwave Characterization of Dielectric Rods

JOSE C. ARANETA, MEMBER, IEEE, MORRIS E. BRODWIN, SENIOR MEMBER, IEEE, AND GREGORY A. KRIEGSMANN

Abstract—**A technique for the simultaneous heating and characterization of dielectric rods using a single microwave source is described. The rod is heated in a rectangular cavity excited by an iris. A variational model for the impedances of homogeneous rods used in the characterization procedure is discussed. It is accurate regardless of the diameter of the rod, even at resonance. Experimental results of β-Al_2O_3 are presented.**

I. Introduction

The CHARACTERIZATION technique to be described is unique in allowing the simultaneous heating and characterization of a dielectric rod while using a single microwave generator. An earlier technique utilized two microwave sources [1].

The inherent speed of microwave heating can result in a significant amount of energy savings and greater throughput of heat-treated rods as compared to conventional heating.

In sintering ceramic rods, the speed of microwave heating makes it possible to discriminate against deleterious slow diffusion processes associated with grain growth [2], [3].

The technique is particularly suitable for processing high-technology ceramics such as β-Al_2O_3, a solid electrolyte used in high-energy density batteries. It can also be used to sinter and characterize high-permittivity ceramics as well as piezoelectric ceramics and ferrites.

Manuscript received November 22, 1983; revised April 19, 1984. This work was supported by the Army Research Office under Contract No. DAAG29-80-K-0031.

J. C. Araneta and M. E. Brodwin are with the Department of Electrical Engineering and Computer Science, Northwestern University, Evanston, IL 60201.

G. A. Kriegsmann is with the Department of Engineering Sciences and Applied Mathematics, Northwestern University, Evanston, IL 60201.

In situ characterization while sintering provides insight into sintering dynamics without the disadvantages of electrodes.

The applicator used to heat and characterize the rod is a rectangular cavity excited by an iris. The rod is mounted in the cavity parallel to the electric-field vector. The dielectric constant and electric conductivity of the rod are deduced by equating the measured admittance of the cavity with the inserted rod with the corresponding admittance derived from the equivalent-circuit representation.

An accurate equivalent circuit representation of the rod is therefore necessary. Marcuvitz [4] gave a variational model for the rod which is accurate only when the rod is very thin compared to the wavelength. It is also invalid near "resonance." Nielsen [5] described a numerical technique which eliminates the limitation on the diameter of the rod. Although Nielsen's method shows an improved representation near resonance, it too suffers a similar deficiency. These models are valid only when the rod is homogeneous, i.e., the electric conductivity and dielectric constant are uniform throughout the rod.

An improved variational model is presented in Section II. It is derived from the same variational formulation attributed to Schwinger [6] that Marcuvitz used. The improved variational model has no restriction on the rod diameter and also yields accurate results in the region of resonance. As compared to the numerical technique of Nielsen [5], the improved variational model is also easier to implement and converges more rapidly. The improvement was realized by using higher order approximations to the variational solution of Schwinger.

The characterization procedure, Section III, involves the equating of the measured and theoretical admittances. This